Riemann Sums - Named after nineteenth century German mathematician Bernhard Riemann.

In section 6.1, the area of under the curve was found by finding the area of a finite number of rectangles (LRAM, RRAM, and MRAM) or a finite number of trapezoids (the Trapezoidal Rule). Every one of these is an example of a ***Riemann Sum***. Let's stay with Rectangles for the time being.

The following steps illustrate what has to happen in order for the sum to be considered a ***Riemann Sum***.

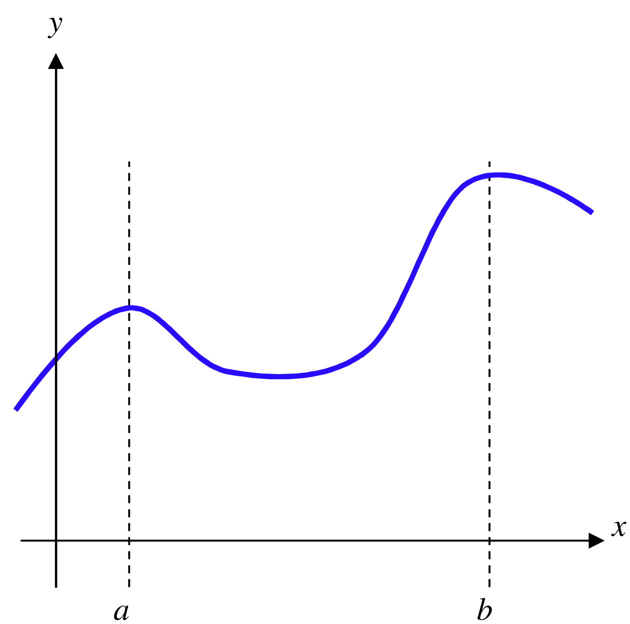
Step 1: Start with a ***continuous*** function on a ***closed interval***.

Step 2: ***Partition*** the interval into n subintervals. The k^{th} subinterval has width Δx_k. (It helps if the subintervals are all the same size, but it really doesn't matter for what we're doing here.)

Step 3: In each subinterval, pick ***any*** number and call the number picked from the k^{th} subinterval c_k (LRAM picked the left endpoint, RRAM picked the right endpoint, and MRAM picked the midpoint).

Step 4: For each interval, using the width, Δx_k of the interval as the base, create a rectangle that extends form the x-axis to the function value, , of the number you picked in each interval. (Note: Some of these rectangles could lie below the x-axis.

Step 5: On each interval, form the product $f(c_k) \bullet \Delta x_k$. (Note: If all the rectangles lie above the x-axis, this would give the area of each rectangle.



Step 6: Find the SUM of each of these products.

$$f(c_1)\bullet \Delta x_1 + f(c_2)\bullet \Delta x_2 + f(c_3)\bullet \Delta x_3 + f(c_4)\bullet \Delta x_4 + \cdots + f(c_n)\bullet \Delta x_n = \sum_{k=1}^{n} f(c_k)\bullet \Delta x_k$$

Following the above steps gives you a ***Riemann Sum* for *f* on the interval [*a*, *b*]**. Every Reimann sum depends on the partition chosen (i.e., the number of subintervals), and the choice of the number within each interval, c_k.

Definite Integral as a Limit of a Riemann Sum

The goal is to develop a Mathematical Definition of a Definite Integral. (What is a Definite Integral you ask?)

While "AREA" is inherently positive, a ***Riemann Sum*** can be negative if the rectangles lie below the x-axis.

A ***Definite Integral*** is defined as a ***Limit of a Riemann Sum***.

Option 1: If you noticed step 2 above, we didn't care if our subintervals were the same width. If we use the notation $\|P\|$ to represent the *longest* subinterval length we can force the longest subinterval length to 0 using a limit the Riemann Sum as follows: $\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\bullet \Delta x_k$

Option 2: If the subintervals are all the same width, we can increase the number of rectangles to infinity using a limit of the Riemann Sum as follows: $\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\bullet \Delta x_k$

Notation for Definite Integrals

The limit notation we used last is the form we will use to develop ***Integral Notation***. As the number of rectangles goes to infinity, the width of each rectangle, Δx , goes to zero. As we did in the section on differentials, we are going to use the notation *dx* to represent this infinitely tiny distance.

The summation notation of sigma is going to be replace with the ***Integral Sign***, $\int$, which looks somewhat like an elongated "*S*" for "S"um.

The $f(c_k)$ which represented a different function value for each interval is going to be replace with $f(x)$ since the *x*-values are going to be so close together it is almost as if we are evaluating the function at every *x*-values in the interval [*a*, *b*]. Combining all of this we have the following notation:

$$\int_a^b f(x)\,dx$$

The notation above is read "***The Integral of f of x from a to b***".

Important (Actually a Theorem): ***IF*** the function is continuous, ***THEN*** the Definite Integral exists. The converse, however, is NOT ALWAYS true.

Using Definite Integrals as Area

We can define the **area under the curve** $\boldsymbol{y = f(x)}$ **from *a* to *b*** as an ***integral*** from ***a*** to ***b*** as long as the curve is nonnegative and integrable on the closed interval [*a*, *b*].

Remember, drawing a picture and using geometry is still a valid method of finding areas.

Example 1. For each of the following examples, sketch a graph of the function, shade the area you are trying to find, then use geometric formulas to evaluate each integral.

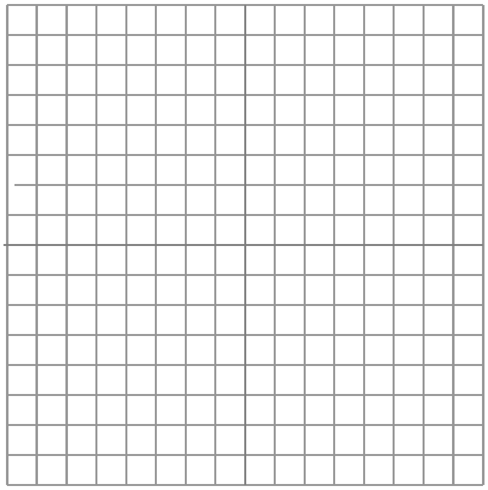
A. $\int_2^8 3\,dx$

B. $\int_{-2}^1 |x|\,dx$

C. $\int_{-3}^3 \sqrt{9-x^2}\,dx$

As we know, "area" is usually positive. But, what happens if the "area" is below the x-axis? A Riemann Sum, and therefore an Integral, can have negative values if the curve lies below the x-axis.

Example 2. Consider the function $f(x) = 3 - x$. Sketch a graph of this function.



A. What is the "Area" between the curve and the x-axis between $x = 4$ and $x = 8$?

B. Evaluate $\int_4^8 (3 - x)\,dx$.

Example 3. Given $\int_0^{\pi} \sin x\, dx = 2$, use what you know about a sine function to evaluate the following integrals.

A. $\int_{\pi}^{2\pi} \sin x\, dx$

B. $\int_0^{2\pi} \sin x\, dx$

C. $\int_0^{\pi/2} \sin x\, dx$

D. $\int_{-\pi}^{\pi} \sin x\, dx$

E. $\int_0^{\pi} (2 + \sin x)\, dx$

Using the fnInt Function on the TI-84

So far, the following major concepts were developed:

1. The Definite Integral was defined as a limit of a Riemann Sum.
2. A Definite Integral can be used to find the Area under if the curve is above the x-axis, and if the curve is below the x-axis the value of the definite integral is "negative area."
3. Since the Definite Integral can be thought of as Area, you can draw a picture and use geometric formulas to find the areas.

But, what happens if you need to find the definite integral of a function that is not a nice geometric shape? For now, you do not have to worry about doing it by hand; you can use your calculator. However, later in this chapter you will learn how to find an integral using new techniques.

How to Use Your Calculator to Find a Definite Integral

Suppose you need to find the area under the curve for a continuous function $f(x)$ on the closed interval $[a,b]$. Here is the basic idea:

1. Press **MATH**
2. Press **9:fnInt(**
3. Fill in the boxes accordingly so it looks like this: $\int_{\boxed{a}}^{\boxed{b}} \boxed{f(x)} d\boxed{x}$. Note that a is the lower limit, b is the upper limit, $f(x)$ is the function, and x is the variable of integration.
4. Press **Enter**
5. Record your result.

Example 4. Evaluate $\int_1^3 \left(x^2 - 2x + 2\right) dx$

Example 5. Evaluate $3 + 2\int_0^{\pi/3} \tan x \, dx$

You can also do the same thing from the graphing screen.

Example 6. Graph $y = \sqrt{x}$ on a standard viewing window. Evaluate $\int_1^8 \sqrt{x} \, dx$.

1. Press **2nd Trace** (which is CALC).
2. Press **7:** $\int f(x)dx$.
3. Enter **Lower Limit** as **1**, enter the **Upper Limit** as **8**.
4. Record your result.

Note: The down side to using this method is the you ***must*** be able to set your window to ***see*** everything.