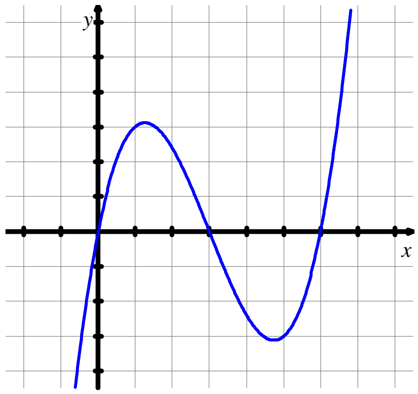
Name:____Per:_________Date:____________________

To complete these notes, go to www.BishSoft.org and watch the video for section 5.3.

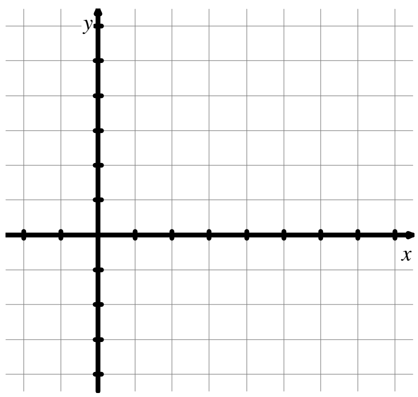
First Derivative Test for Extrema

We have already determined that relative extrema occur at critical points. The behavior of the first derivative before and after those critical points will help determine whether or not the function has a relative maximum or minimum (or neither) at these critical points.

Example 1. Given the graph of f below, label the relative extrema.



Example 2. Sketch f' as accurately as you can on the graph below. Label the x-values of the extrema from f on f'.



Check #1 and #2 in the box below with the graph you just sketched.

The First Derivative Test

Let f be a continuous function, and let c be a critical point.

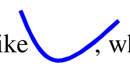
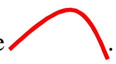
1. If f' changes sign from positive to negative at c, then f has a local maximum value at c.
2. If f' changes sign from negative to positive at c, then f has a local minimum value at c.
3. If f' **DOES NOT** change signs, then there is no local extreme value at c.

Note: If you are asked to find the absolute maximum (or just a maximum) of a function on a closed interval, you **MUST** test the endpoints also, and it may be just as simple to plug in the endpoints and the critical points.

Example 3. Find where the function $h(x)=x^4-4x^3$ increasing and decreasing, then use the first derivative test to determine any local extrema.

Example 4. Find there the function $g(x)=x^2e^x$ is increasing and decreasing, then find any local extrema and absolute extrema.

Concavity

Concavity deals with how a graph is curved. A graph that is **Concave Up** looks like , while a graph that is **Concave Down** looks like . We can use the **Second Derivative** to determine the concavity of a function.

Definition of Concavity

Let $y=f(x)$ be a differentiable function on an interval *I*. The graph of $f(x)$ is **Concave Up** on *I* if f' is increasing on *I*, and **Concave Down** on *I* if f' is decreasing on *I*.

If the first derivative is increasing, then the second derivative must be _________________.

If the first derivative is decreasing, then the second derivative must be _________________.

Thus, instead of using the definition of concavity to determine whether the function is concave up or down, we can use the following test.

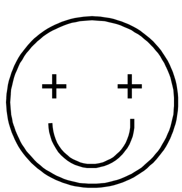
Concavity Test

The graph of a twice-differentiable function $y=f(x)$ is

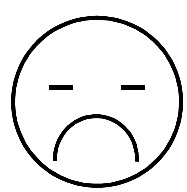
Concave Up on any interval where $y''>0$, and

Concave Down on any interval where $y''<0$.

The Concavity Test can be summed up by the following pictures. While this is a humorous (and hopefully helpful) way to remember concavity, please understand that this is **NEVER** to be used as a justification on **ANY** test!

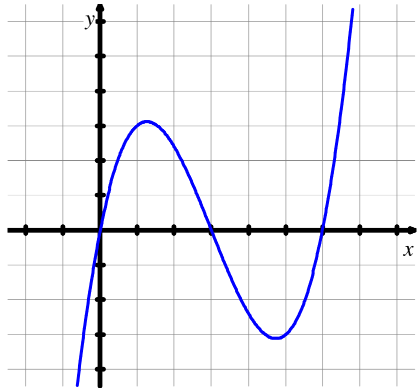


f'' positive $\Rightarrow$ Concave Up



f'' negative $\Rightarrow$ Concave Down

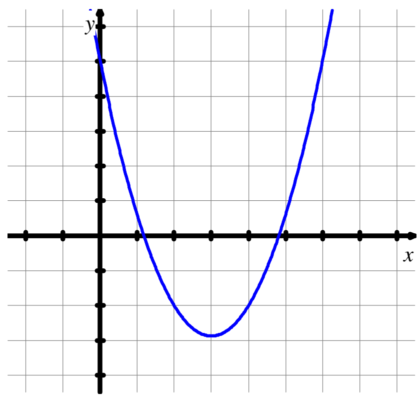
Example 5. Using the same graph as our previous example, indicate which portions of the graph are concave up, and which portions are concave down. Label the point where the graph changes concavity.



Example 6. The graph of f' is graphed below.

A. Label the x-value where the graph changes concavity. Verify this with the definition of concavity as it relates to the first derivative.

B. Sketch the graph of f'' on the graph from part (**A**) above. Verify this with the definition of concavity as it relates to the first derivative.



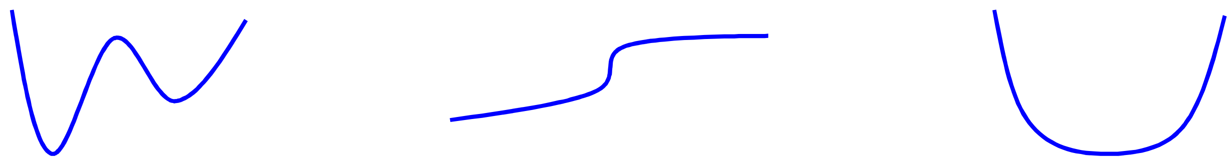
Example 7. Find the intervals where the function $g(x) = x^4 - 4x^3$ is concave up and concave down.

Points of Inflection

Definition

A point where the graph of a function **has a Tangent Line** (even if it's a vertical tangent line) **AND** where **the Concavity Changes** is a **Point of Inflection.**

Example 8. Using each picture, estimate each point of inflection, if any, and sketch the tangent line at that point.



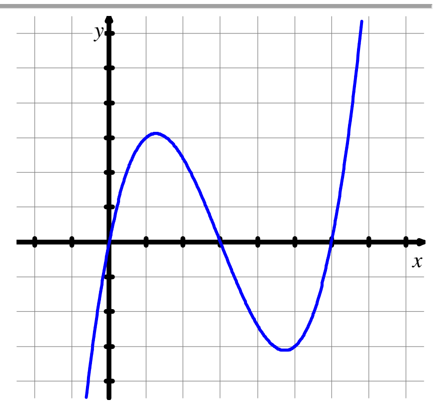
Since points of inflection occur when the graph changes ____________________________, and a graph changes concavity when the ____ changes from positive to negative (or vice-versa), then if we wanted to find the points of inflection of a graph, we only need to focus on when the second derivative equals 0 (or does not exist).

NOTE: Just because the second derivative equals zero (or does not exist) you are **NOT** guaranteed that the function has a point of inflection. The second derivative MUST change signs (meaning concavity changed) in order for a point of inflection to exist!

Example 9. Find the points of inflection of $g(x) = 3x^5 + 10x^4 + 15x + 7$.

Example 10. To the right is the function from Example 1. Look at the point where the function has a relative maximum. Is the graph concave up or concave down at that point?

Now look at the point where the function has a relative minimum. Is the graph concave up or concave down at that point?



As long as the function is twice-differentiable (meaning the first derivative is a smooth curve), then we can actually determine whether or not a **Critical Point** is a relative maximum or minimum **WITHOUT** testing values to the right and left of the point. We can use the **Second Derivative Test**.

Second Derivative Test for Local Extrema

If $f'(c)=0$ (which makes $x = c$ a critical point), **AND** $f''(c)<0$, then f has a local **MAXIMUM** at $x = c$.

If $f'(c)=0$ (which makes $x = c$ a critical point), **AND** $f''(c)>0$, then f has a local **MINIMUM** at $x = c$.

NOTE: If the second derivative is equal to zero (or undefined) then the **Second Derivative Test** is **INCONCLUSIVE**. Also remember, this is all for **Relative Maximums** and **Relative Minimums**. Most free response questions on tests focus on **Absolute Extrema**. So, be sure to test **ALL** candidates found from the first derivative.

Remember the happy (and sad) faces? If a critical point happens to occur in an interval where the graph of the function is **CONCAVE UP**, then that critical point is a relative **MINIMUM**. If a critical point happens to occur in an interval where the graph of the function is **CONCAVE DOWN**, then that critical point is a relative **MAXIMUM**.

Example 11. Use the Second Derivative Test to identify any relative extrema for the function $g(x)=-x^4+4x^3-4x^2+1$.

NOTES FOR OPTIMIZATION AND ABSOLUTE EXTREMA PROBLEMS

Whenever you are required to Maximize or Minimize a function, you MUST **justify** whether or not your answer is actually a maximum or a minimum. You may use the **First Derivative Test** (testing points to the left and right of the critical points in the first derivative to see if the sign of the first derivative changes from positive to negative or vice-versa), or the **Second Derivative Test** (plugging in the critical points to the second derivative to see if the critical points occur when the original function was concave up or down).

ALWAYS REMEMBER that both of these tests are checking for relative extrema. If you have a **CLOSED** interval, you must check the endpoints to make sure the absolute maximum or minimum values do not happen to occur there. If you have a closed interval, it is best just to check **ALL** critical points and endpoints.

A Quick Summary of Sections 5.2 and 5.3